# Pareto efficiency and competitive equilibrium

Consider an exchange economy with two goods and two agents whose preferences are determined by the utility function

$$u^i(x, y) = x^\alpha y^{1-\alpha}, \quad i = 1, 2, \quad 0 < \alpha < 1$$

and the total resources are

$$\omega^1 + \omega^2 = (10, 10)$$

1. Prove that the allocation

   $$x_1 = x_2 = y_1 = y_2 = 5$$

   is Pareto efficient.

2. Determine initial endowments $\omega^1, \omega^2$ with $\omega^1 \neq \omega^2$ such that, in this economy, the previous allocation is a competitive equilibrium.

## Solution

1. Pareto efficient allocations:

   We will use the following preferences:

$$v_1(x_1, y_1) = \alpha \ln x_1 + (1-\alpha) \ln y_1 = \ln u_1$$
$$v_2(x_2, y_2) = \alpha \ln x_2 + (1-\alpha) \ln y_2 = \ln u_2$$

   The Pareto efficient allocations are the solution of

$$\max \; t(\alpha \ln x_1 + (1-\alpha) \ln y_1) + (1-t)(\alpha \ln x_2 + (1-\alpha) \ln y_2)$$
$$\text{s.t.} \quad x_1 + x_2 = \omega_1^1 + \omega_1^2$$
$$y_1 + y_2 = \omega_2^1 + \omega_2^2$$

   The first-order conditions are:

$$\frac{\partial L}{\partial x_1} : \frac{t\alpha}{x_1} = \lambda$$
$$\frac{\partial L}{\partial x_2} : \frac{(1-t)\alpha}{x_2} = \lambda$$
$$\frac{\partial L}{\partial y_1} : \frac{t(1-\alpha)}{y_1} = \mu$$
$$\frac{\partial L}{\partial y_2} : \frac{(1-t)(1-\alpha)}{y_2} = \mu$$

   From the first two, we get:

$$t\alpha = \lambda x_1$$
$$(1-t)\alpha = \lambda x_2$$

   Adding these, we obtain:

$$t\alpha + (1-t)\alpha = \lambda(x_1 + x_2)$$

$$t\alpha + (1-t)\alpha = \lambda\omega_1$$

   Therefore,

$$\lambda = \frac{t\alpha + (1-t)\alpha}{\omega_1}$$

   Similarly, from the last two,

$$t(1-\alpha) = \mu y_1$$
$$(1-t)(1-\alpha) = \mu y_2$$

   Adding these, we get:

$$t(1-\alpha) + (1-t)(1-\alpha) = \mu(y_1 + y_2)$$

$$t(1-\alpha) + (1-t)(1-\alpha) = \mu\omega_2$$

   Thus,

$$\mu = \frac{\omega_2}{t(1-\alpha) + (1-t)(1-\alpha)}$$

The Pareto efficient allocations are:

$$x_1 = \frac{t\alpha}{\lambda} = \frac{t\alpha}{\frac{t\alpha+(1-t)\alpha}{\omega_1}} = \frac{t}{t+(1-t)}\omega_1 = \omega_1 t$$

$$x_2 = \frac{(1-t)\alpha}{\lambda} = \frac{(1-t)\alpha}{\frac{t\alpha+(1-t)\alpha}{\omega_1}} = \frac{(1-t)}{t+(1-t)}\omega_1 = (1-t)\omega_1$$

$$y_1 = \frac{t(1-\alpha)}{\mu} = \frac{t(1-\alpha)}{\frac{\omega_2}{t(1-\alpha)+(1-t)(1-\alpha)}} = \frac{t}{t+(1-t)}\omega_2 = \omega_2 t$$

$$y_2 = \frac{(1-t)(1-\alpha)}{\mu} = \frac{(1-t)(1-\alpha)}{\frac{\omega_2}{t(1-\alpha)+(1-t)(1-\alpha)}} = \frac{(1-t)}{t+(1-t)}\omega_2 = (1-t)\omega_2$$

where $0 \leq t \leq 1$.

Taking $t = \frac{1}{2}$, we obtain :

$$x_1 = (1/2)(x_1 + x_2) = 10/2 = 5$$

$$x_2 = (1-1/2)(x_1 + x_2) = 10/2 = 5$$

$$y_1 = 1/2(y_1 + y_2) = 10/2 = 5$$

$$y_2 = (1-1/2)(y_1 + y_2) = 10/2 = 5$$

Which is PE

2. The first-order conditions (FOCs) for agent 1 are:

$$\nabla u_1(5,5) = \lambda(p_1, p_2)$$

that is,

$$\frac{\alpha}{x_1} = \lambda p_1$$

$$\frac{1-\alpha}{y_1} = \lambda p_2$$

with $x_1 = y_1 = 5$

$$\frac{\alpha}{5} = \lambda p_1$$

$$\frac{1-\alpha}{5} = \lambda p_2$$

Therefore,

$$\frac{p_1}{p_2} = \frac{\alpha}{1-\alpha}$$

The first-order conditions (FOCs) for agent 2 are:

$$\nabla u_2(5,5) = \mu(p_1, p_2)$$

that is,

$$\frac{\alpha}{x_2} = \mu p_1$$
$$\frac{1-\alpha}{y_2} = \mu p_2$$

with $x_2 = y_2 = 5$

$$\frac{\alpha}{5} = \mu p_1$$
$$\frac{1-\alpha}{5} = \mu p_2$$

Therefore,

$$\frac{p_1}{p_2} = \frac{\alpha}{1-\alpha}$$

We can take $p_1 = \alpha$ and $p_2 = 1 - \alpha$. And

$$\omega^1 = (\omega_1^1, \omega_1^2)$$
$$\omega^2 = (10 - \omega_1^1, 10 - \omega_1^2)$$

Let's see if this verifies the budget constraints

$$p_1 x_1 + p_2 y_1 = \alpha\omega_1^1 + p_2\omega_2^1$$
$$p_1 x_2 + p_2 y_2 = \alpha\omega_1^2 + p_2\omega_2^2$$

$$\alpha 5 + (1-\alpha)5 = \alpha\omega_1^1 + (1-\alpha)\omega_1^2$$
$$\alpha 5 + (1-\alpha)5 = \alpha(10 - \omega_1^1) + (1-\alpha)(10 - \omega_1^2)$$

$$5 = \alpha\omega_1^1 + (1-\alpha)\omega_1^2$$
$$5 = \alpha(10 - \omega_1^1) + (1-\alpha)(10 - \omega_1^2)$$

See for example $\omega_1^1 = 3$. Then

$$5 = \alpha 3 + (1-\alpha)\omega_1^2$$

$$\omega_1^2 = \frac{5-3\alpha}{1-\alpha}$$

Let's check if budget constraint is satisfied for agent 2:

$$5 = \alpha(7) + (1-\alpha)(10 - \frac{5-3\alpha}{1-\alpha})$$

$$5 = \alpha(7) + 10 - \frac{5-3\alpha}{1-\alpha} - 10\alpha + \alpha\frac{5-3\alpha}{1-\alpha}$$

$$-5 = -3\alpha - \frac{5-3\alpha}{1-\alpha} + \alpha\frac{5-3\alpha}{1-\alpha}$$
$$-5 = -3\alpha - (1-\alpha)\frac{5-3\alpha}{1-\alpha}$$
$$-5 = -3\alpha - 5 + 3\alpha$$
$$-5 = -5$$